AGT conjecture and Integrable structure of Conformal field theory for $c = 1$

A. Belavin

Landau Institute for Theoretical Physics, RAS, Chernogolovka, Russia

and

V. Belavin

Theory Department, Lebedev Physical Institute, RAS, Moscow, Russia

Abstract

AGT correspondence gives an explicit expressions for the conformal blocks of $d = 2$ conformal field theory. Recently an explanation of this representation inside the CFT framework was given through the assumption about the existence of the special orthogonal basis in the module of algebra $\mathcal{A} = Vir \otimes \mathcal{H}$. The basis vectors are the eigenvectors of the infinite set of commuting integrals of motion. It was also proven that some of these vectors take form of Jack polynomials. In this note we conjecture and verify by explicit computations that in the case of the Virasoro central charge $c = 1$ all basis vectors are just the products of two Jack polynomials. Each of the commuting integrals of motion becomes the sum of two integrals of motion of two noninteracting Calogero models. We also show that in the case $c \neq 1$ it is necessary to use two different Feigin-Fuks bosonizations of the Virasoro algebra for the construction of all basis vectors which take form of one Jack polynomial.

1 Introduction

AGT conjecture [1] reveals a deep connection between 2d CFT and $N = 2$ SUSY gauge theories. This correspondence turns out to be very important for the 2d CFT. In particular, it gives a remarkable explicit representation for the conformal block coefficients in terms of the Nekrasov partition functions [2]. This representation was not known in the framework of 2d CFT. Certainly, it is a challenge for CFT. It is necessary to achieve appropriate understanding of this new representation for conformal blocks staying inside CFT frame. Different attempts to derive the Nekrasov representation for the conformal blocks using the conformal bootstrap approach [3, 4] have been performed recently [5–7]. Many new features, as well as some new connections between Seiberg-Witten theory, Matrix Models,

Dotsenko-Fateev representation [8] for conformal blocks, Selberg integrals, *etc*, were found (see e.g. [9–12]).

Recently an important for the understanding of the new representation for conformal blocks step was made in [13–15]. It was assumed by Alday and Tachikawa in [13] that in the Hilbert space $\mathcal{H}_a$, formed by the tensor product of the Fock module, which corresponds to the so called $U(1)$ factor [1], and the Virasoro module V_a a special orthogonal basis which consists of $|\vec{Y}\rangle$ for all pairs $\vec{Y}$ of Young diagrams can be constructed. It was also assumed in [13], that the matrix elements of the product of Carlsson and Okounkov vertex operator [16] and the Virasoro primary field between two vectors from two different copies of $\mathcal{H}_a$ have a very simple factorized form. These matrix elements coincide with the Nekrasov partition functions for the bifundamental hypermultiplet. In [15] some evidence of this proposal based on the explicit computations was given. It was also proven in [15] that if first of two Young diagrams is empty the corresponding basis vector has form of Jack polynomial [17]. To get the Jack representation for such vectors it is necessary to use the Feigin-Fuks bosonization of the Virasoro algebra[1]. It was assumed and verified in [15] that this basis diagonalizes an infinite set of commuting Integrals of Motion of some integrable hierarchy. However, the explicit form of the vectors of the orthogonal basis in the general case remains still unknown.

In this paper we continue to study this basis and the underlying integrable structure. First of all we want to understand the form of the vectors for both particular cases when one of Young diagrams, the first or the second one, is empty. In addition to the basis vectors $|Y, \varnothing\rangle$ constructed in [15] we construct the vectors of the form $|\varnothing, Y\rangle$. Their form is rather complicated in terms of the Feigin-Fuks bosonization used for constructing vectors $|Y, \varnothing\rangle$. However, once the second possible bosonization is used, they simplify and also take the form of Jack polynomials. The naive assumption that if both Young diagrams are not empty the vectors are the products of two Jack polynomials which depend on two different Feigin-Fuks bosons is not correct. However it is true in the case $c = 1$. In this case the bosons of two different sets commute if their indices are of the same sign. We verified this statement by explicit computations. This facts lead us to the conjecture that the Integrals of motion of [15] for $c = 1$ coincide with the sum of Integrals of Motion of two noninteracting Calogero models. This statement was also verified using the explicit expressions given in [15] for the first Integrals of motion.

The paper is organized as follows. In section 2 we recall the definition of the conformal block functions of 2d CFT and describe their relation to Nekrasov's partition functions. In section 3 we discuss the two possible versions of Feigin-Fuks bosonization of the Virasoro algebra and the notion of Liouville reflection operator [21]. This is important for the correct choice of the arguments of Jack functions in some special cases, as well as in the formulation of the conjecture about the form of basis vectors for $c = 1$. In section 4 we focus our attention on the case of central charge $c = 1$. The Jack functions are reduced in this case to the Schur polynomials. We conjecture that in this case all basis vectors are products of two Schur polynomials and verify this statement up to the level 3. Some formulas related to the Nekrasov representation of the conformal block, the definition of the Jack symmetric functions and their basic properties are collected in the appendices.

---

[1]For other connection between the Jack symmetric polynomials and the Virasoro algebra see [18].

2 AGT correspondence and a special basis of states in the highest weight representations of $Vir \otimes \mathcal{H}$

Conformal blocks [3] are special analytic functions on a Riemann surface of genus g with n punctures. They play an important role in $d = 2$ CFT. The AGT conjecture [1] provides us with explicit expressions for the conformal blocks in terms of the Nekrasov partition functions of a certain class of $N = 2$ SCFTs [19]. Below we shall consider the case of the 4-point conformal block on the sphere. In addition to the four-point projective invariant q it depends on six parameters: the central charge c, four external conformal dimensions Δ_i and the dimension of the intermediate primary filed Φ_Δ (for details see Appendix A). It can be represented by the following diagram

$$(2.1)$$
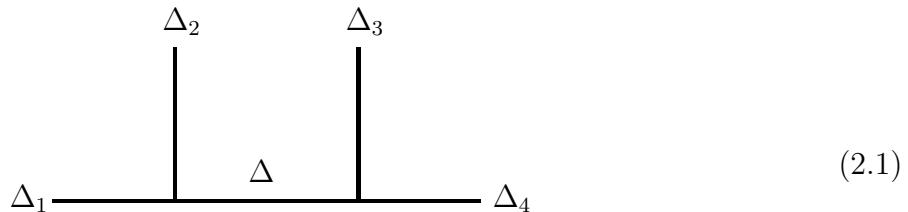
which defines one of three possible channels of fusing external fields into intermediate one. Instead of the conformal dimensions Δ_i, Δ, we use sometimes the parameters λ_i, P:

$$\Delta_i = Q^2/4 - \lambda_i^2, \qquad \Delta = Q^2/4 - P^2, \tag{2.2}$$

while, instead of Virasoro central charge we introduce the parameter b:

$$c = 1 + 6Q^2, \qquad Q = 1/b + b. \tag{2.3}$$

In terms of new variables the $4-$point conformal block is denoted as $F^{\mathsf{V}}(q, \lambda_i, P)$. Due to the operator product expansion it has the form of a power series

$$F^{\mathsf{V}}(q, \lambda_i, P) = \sum_{N=0}^{\infty} {}^{\mathsf{V}}\langle N; \lambda_1, \lambda_2 | N; \lambda_3, \lambda_4 \rangle^{\mathsf{V}} q^N, \tag{2.4}$$

where $|N; \lambda_1, \lambda_2\rangle^{\mathsf{V}}$ stands for the so-called chain vector, which is defined as a linear combination of Nth level descendants $L_{-k_1} L_{-k_2} \ldots |P\rangle$ (L_n being the generators of the Virasoro algebra). Conformal symmetry determines uniquely this function [3]. It implies that

$$L_n | N; \lambda_1, \lambda_2 \rangle^{\mathsf{V}} = (\Delta + n\Delta_1 - \Delta_2 + N - n) | N - n; \lambda_1, \lambda_2 \rangle^{\mathsf{V}}, \tag{2.5}$$

for any positive n. To formulate the AGT conjecture we introduce function

$$F(q) = F^{\mathcal{H}}(q) F^{\mathsf{V}}(q), \tag{2.6}$$

where $F^{\mathcal{H}}(q)^2$ stands for the Heisenberg conformal block

$$F^{\mathcal{H}}(q) = (1 - q)^{2(\frac{Q}{2} + \lambda_1)(\frac{Q}{2} - \lambda_3)}. \tag{2.7}$$

---

[2]To simplify notation, we will suppress sometimes some of the arguments of the conformal blocks.

The term "Heisenberg" for $F^{\mathcal{H}}(q)$ is related to the following interpretation of this function. Let a_n be generators of the Heisenberg algebra $\mathcal{H}$ with the commutation relations

$$[a_n, a_m] = \frac{n}{2}\delta_{n+m,0}. \tag{2.8}$$

The vectors $|N, \alpha\rangle^{\mathcal{H}}$ in the module $\mathcal{H}$ with the highest vector $|0\rangle$ (i.e. $a_n|0\rangle = 0$ for $n > 0$), is defined as the only linear combination of vectors $|\vec{k}\rangle = a_{-1}^{k_1} a_{-2}^{k_2} \dots |P\rangle$ on the level N that obeys the recursive relation

$$a_n|N, \alpha\rangle^{\mathcal{H}} = \alpha|N - n, \alpha\rangle^{\mathcal{H}} \tag{2.9}$$

One can evaluate explicitly

$$|N, \alpha\rangle^{\mathcal{H}} = \sum_{k_1, k_2, \dots} \prod_{l=1}^{\infty} \frac{(2\alpha)^{k_l}}{k_l! \, l^{k_l}} |\vec{k}\rangle \tag{2.10}$$

Once the conjugation $(a_k)^+ = a_{-k}$ is defined, the Heisenberg conformal block is expressed in terms of the vectors $|N, \alpha\rangle^{\mathcal{H}}$

$$F^{\mathcal{H}}(q) = \sum_N {}^{\mathcal{H}}\langle N, i(\frac{Q}{2} - \lambda_3)|N, i(\frac{Q}{2} + \lambda_1)\rangle^{\mathcal{H}} \, q^N. \tag{2.11}$$

Now one can build the "mixed" block $F(q)$ from the vectors $|N, \lambda_1, \lambda_2\rangle$, which belong to the $\mathcal{A}$-module, the representation space of the algebra $\mathcal{A} = Vir \otimes \mathcal{H}$. (The $\mathcal{A}$-module is just the tensor product of the $\mathcal{H}$– and of the Virasoro–modules)

$$F(q) = \sum_N \langle N, \lambda_3, \lambda_4 | N, \lambda_1, \lambda_2\rangle \, q^N, \tag{2.12}$$

where

$$|N, \lambda_1, \lambda_2\rangle = \sum_{\substack{N_1, N_2 \\ N_1 + N_2 = N}} |N_1, i\lambda_1\rangle^{\mathcal{H}} |N_2; \lambda_1, \lambda_2\rangle^{\vee}. \tag{2.13}$$

In these notation the AGT conjecture claims that

$$\langle N, \lambda_3, \lambda_4 | N, \lambda_1, \lambda_2\rangle = \sum_{\substack{Y_1, Y_2 \\ |Y_1| + |Y_2| = N}} \frac{Z_f(\vec{a}, \vec{Y}, \mu_1) Z_f(\vec{a}, \vec{Y}, \mu_2) Z_{af}(\vec{a}, \vec{Y}, \mu_3) Z_{af}(\vec{a}, \vec{Y}, \mu_4)}{D(\vec{a}, \vec{Y}) \bar{D}(\vec{a}, \vec{Y})}. \tag{2.14}$$

Here, in the right hand side the summation runs over pairs of Young tableaux (Y_1, Y_2) with the total number of cells equal to N. The explicit form of $Z_f(\vec{a}, \vec{Y}, \mu)$, $Z_{af}(\vec{a}, \vec{Y}, \mu)$ and $D(\vec{a}, \vec{Y})$, $\bar{D}(\vec{a}, \vec{Y})$ can be found in Appendix A. The parameters of Nekrasov's partition function are related to the parameters of the conformal block as follows:

$$\begin{aligned}
\mu_1 &= \frac{Q}{2} - (\lambda_1 + \lambda_2), & \mu_2 &= \frac{Q}{2} - (\lambda_1 - \lambda_2), \\
\mu_3 &= \frac{Q}{2} - (\lambda_3 + \lambda_4), & \mu_4 &= \frac{Q}{2} - (\lambda_3 - \lambda_4),
\end{aligned} \tag{2.15}$$

and

$$\vec{a} = (a, -a), \qquad a = P. \tag{2.16}$$

4

The form of the scalar product (2.14) as a sum of simple rational Nekrasov functions leads to the natural idea [14, 15] that the vector $|N\rangle$ can be written as a linear combination of some orthogonal vectors $|Y_1, Y_2\rangle$ which form a basis in the $\mathcal{A}$-module. To ensure (2.14), the expansion of $|N\rangle$ over this basis should be

$$|N, \lambda_1, \lambda_2\rangle = \sum_{\vec{Y}, |\vec{Y}|=N} \frac{Z_f(\vec{a}, \vec{Y}, \mu_1) Z_f(\vec{a}, \vec{Y}, \mu_2)}{D(\vec{a}, \vec{Y})} |\vec{Y}\rangle. \tag{2.17}$$

The basis vectors up to the level 6 and of the form $|Y_1, \varnothing\rangle$ on the general level were calculated in [15]. When one of Young tableaux is empty the corresponding vector is expressed through the Jack polynomials.

To write down the known basis vectors [15] explicitly, as well as to formulate the conjecture about the form of vectors $|Y_1, Y_2\rangle$ in the special case $c = 1$, it is necessary to use the Feigin-Fuks representation for the Virasoro modules. It will be done in the next section.

3 Two bosonizations of Virasoro algebra and Liouville reflection operator

Let us consider the Heisenberg algebra with a set of generators b_k and $\hat{P}$, commuting as

$$[b_n, b_m] = \frac{n}{2} \delta_{n+m,0}, \qquad [b_n, \hat{P}] = 0. \tag{3.1}$$

The Feigin-Fuks representation [20] for Virasoro algebra is

$$L_n = \sum_{k \neq 0, n} b_k b_{n-k} + i(nQ + 2\hat{P})b_n,$$

$$L_0 = 2 \sum_{k>0} b_{-k} b_k + \frac{Q^2}{4} - \hat{P}^2. \tag{3.2}$$

Fock space with the vacuum vector $|P\rangle$, such that $b_k|P\rangle = 0$ for $k > 0$ and $\hat{P}|P\rangle = P|P\rangle$, is the Virasoro highest weight representation with the conformal dimension $\Delta = \frac{Q^2}{4} + P^2$ and the central charge $c = 1 + 6Q^2$. A second set of the generators, which we denote b_k^{R}, exist in the universal enveloping Heisenberg algebra. It is connected with the previous one by some unitary transform $\hat{S}(P)$

$$b_k^{\mathrm{R}} = \hat{S}(P) \, b_k \, \hat{S}^{-1}(P). \tag{3.3}$$

The requirement, which fixes the new generators b_k^{R}, is that they should give the second possible Feigin-Fuks representation of the Virasoro algebra

$$L_n = \sum_{k \neq 0} b_k^{\mathrm{R}} b_{n-k}^{\mathrm{R}} + i(nQ - 2\hat{P})b_n^{\mathrm{R}}. \tag{3.4}$$

The operator $\hat{S}(P)$ is called reflection operator [21]. It plays some special role in Liouville field theory. In general, this is some nonlinear transformation which is unknown in the closed

5

form. However, the first terms of the formal expansion can be constructed explicitly. Say, for b^{R}_{-1}, b^{R}_{-2}, we have the following system of the constraints

$$
\begin{aligned}
L_{-1}(b^{\mathrm{R}}_k, -\hat{P})|P\rangle &= L_{-1}(b_k, \hat{P})|P\rangle, \\
L^2_{-1}(b^{\mathrm{R}}_k, -\hat{P})|P\rangle &= L^2_{-1}(b_k, \hat{P})|P\rangle, \\
L_{-2}(b^{\mathrm{R}}_k, -\hat{P})|P\rangle &= L_{-2}(b_k, \hat{P})|P\rangle,
\end{aligned}
\tag{3.5}
$$

which should be considered as the equations for unknown b^{R}_{-1} and b^{R}_{-2}. Then the first terms of the series for b^{R}_{-1}, b^{R}_{-2} are

$$
b^{\mathrm{R}}_{-1} = \frac{Q-2p}{Q+2p} b_{-1} + \frac{4iPQ}{(2P-Q)(1+4P^2+6PQ+2Q^2)} b_{-2}b_1 - \\
- \frac{16PQ}{(2P+Q)(2P-Q)(1+4P^2+6PQ+2Q^2)} b^2_{-1}b_1 + \dots,
\tag{3.6}
$$

$$
b^{\mathrm{R}}_{-2} = -\frac{2P+8P^3-Q-6PQ^2-2Q^3}{(2P+Q)(1+4P^2+6PQ+2Q^2)} b_{-2} + \frac{8iPQ}{(2P+Q)(1+4P^2+6PQ+2Q^2)} b^2_{-1} + \dots.
\tag{3.7}
$$

The standard form of the Virasoro conjugation $L^+_n = L_{-n}$ fixes the conjugation of the bosonizing Heisenberg generators

$$
b^+_n = -b_{-n}, \qquad \hat{P}^+ = -\hat{P}.
\tag{3.8}
$$

As we mentioned above, two different bosonizations can be used in constructing chain vectors (2.13). Moreover, both sets of Feigin-Fuks generators b_k and b^{R}_k turn out to be relevant. For example, if one of Young tableaux Y_2 or Y_1 is empty, then, using a_k and b_k generators, one finds

$$
|Y, \varnothing\rangle = \hat{J}^{-1/b^2}_Y(a_{-1}+b_{-1}, a_{-2}+b_{-2}, \dots)|P\rangle,
\tag{3.9}
$$

where the polynomials $\hat{J}^\alpha_Y$ are related with the Jack polynomials (see Appendix B) as follows

$$
\hat{J}^\alpha_Y(p_1, p_2, \dots) = J^\alpha_Y(x_1, x_2, \dots)
\tag{3.10}
$$

and $p_k = (\alpha)^{1/2}(x^k_1 + \dots + x^k_{|Y|})$. In the same time, vectors $|\varnothing, Y\rangle$ have rather complicated form in terms of a_k, b_k. While the second bosonization (3.4) gives the following simple expression

$$
|\varnothing, Y\rangle = \hat{J}^{-1/b^2}_Y(a_{-1}+b^{\mathrm{R}}_{-1}, a_{-2}+b^{\mathrm{R}}_{-2}, \dots)|P\rangle.
\tag{3.11}
$$

4 The orthogonal basis for the case $c = 1$

As we have seen in the previous section the basis vectors have rather simple form if one of Young tableaux is empty. But the general answer, when both Young tableaux are non-trivial, remains uncertain. It turns out that the structure of the orthogonal basis can be clarified completely in the case of central charge $c = 1$. The main problem encountered in the previous analysis is that in general, one have to deal with the two non-commuting sets of generators $a_k + b_k$ and $a_k + b^{\mathrm{R}}_k$. The orthogonality of the basis vectors with one

6

empty diagram is obvious, given that they correspond to the Jack symmetric polynomials, depending on either one, or another set of generators. But the naive conjecture that for both non empty Young tableaux the vectors $|Y_1, Y_2\rangle$ are proportional to the product of two Jack polynomials, where the first polynomial depends on $a_k + b_k$ and the second one depends on $a_k + b_k^{\rm R}$, can not be correct because of the above mentioned non-commutativity.

For $c = 1$ the situation becomes more simple. The reflection relation reduces to the flip of sign $b_k^{\rm R} = -b_k$. Therefore the commutativity between Jack polynomials, corresponding to two different camps of generators $a_k + b_k$ and $a_k - b_k$ just follows from the fact that

$$[a_k + b_k, a_k - b_k] = 0. \tag{4.1}$$

Thus, there exists a natural orthogonal basis in $\mathcal{A}$ in this case[3]

$$|Y_1, Y_2\rangle = \hat{J}_{Y_1}(a_k + b_k)\hat{J}_{Y_2}(a_k - b_k)|P\rangle \tag{4.2}$$

Now we are going to verify that the expansion of the chain vectors $|N\rangle$ (2.13) in this basis leads to the AGT representation of the conformal block for $c = 1$. Using orthogonality the coefficients of the vector $|N\rangle$ in the basis $\{|Y_1, Y_2\rangle\}$

$$|N\rangle = \sum_{Y_1, Y_2} C_N(Y_1, Y_2)|Y_1, Y_2\rangle \tag{4.3}$$

are given in the form

$$C_N(Y_1, Y_2) = \frac{\langle Y_1, Y_2|N\rangle}{\langle Y_1, Y_2|Y_1, Y_2\rangle}. \tag{4.4}$$

Therefore one have to verify that

$$C_N(Y_1, Y_2) = \frac{Z_f(\vec{a}, \vec{Y}\mu_1), Z_f(\vec{a}, \vec{Y}, \mu_2)}{D(\vec{a}, \vec{Y})}, \tag{4.5}$$

where the parameters are defined in (2.15) and (2.16).

Level 1: At level one, the chain vector is

$$|1\rangle^{\mathsf{V} \otimes \mathcal{H}} = |1\rangle^{\mathsf{V}} + |1\rangle^{\mathcal{H}} = \beta_1^{\mathsf{V}} L_{-1}|P\rangle + \beta_1^{\mathcal{H}} a_{-1}|P\rangle, \tag{4.6}$$

where the coefficients are easily found from (2.5) and (2.9)

$$\beta_1^{\mathsf{V}} = \frac{\Delta + \Delta_1 - \Delta_2}{2\Delta}, \qquad \beta_1^{\mathcal{H}} = i\lambda_1. \tag{4.7}$$

The basis vectors are expressed in terms of Jack/Schur symmetric polynomials

$$\begin{aligned}
|\{1\}, \varnothing\rangle &= \hat{J}_{\{1\}}^{(1)}(a_k - b_k)\hat{J}_{\varnothing}^{(1)}(a_k + b_k)|P\rangle = (a_{-1} - b_{-1})|P\rangle, \\
|\varnothing, \{1\}\rangle &= \hat{J}_{\varnothing}^{(1)}(a_k - b_k)\hat{J}_{\{1\}}^{(1)}(a_k + b_k)|P\rangle = (a_{-1} + b_{-1})|P\rangle,
\end{aligned} \tag{4.8}$$

---

[3]Here and below we use short notation $\hat{J}_Y$ for $\hat{J}_Y^{(\alpha=1)}$.

Using Feigin-Fucks bosonization the first few generators are

$$L_{-1} = 2i\hat{P}b_{-1} + 2b_{-2}b_1 + 2b_{-3}b_2 + \dots,$$
$$L_{-2} = 2i\hat{P}b_{-2} + b_{-1}^2 + 2b_{-3}b_1 + \dots,$$
$$L_{-3} = 2i\hat{P}b_{-3} + 2b_{-1}b_{-2} + 2b_{-3}b_1 + \dots.$$

(4.9)

One can find the scalar products of the chain vector with the basis vectors on the level one

$$\langle\{1\}, \varnothing|1\rangle^{V\otimes\mathcal{H}} = -\frac{i(P - \lambda_1 - \lambda_2)(P - \lambda_1 + \lambda_2)}{2P},$$
$$\langle\varnothing, \{1\}|1\rangle^{V\otimes\mathcal{H}} = \frac{i(P + \lambda_1 - \lambda_2)(P + \lambda_1 + \lambda_2)}{2P},$$

(4.10)

This coincides with the r.h.s. of (4.5) which looks in this case as follows

$$\frac{Z_f(\vec{a}, \vec{Y}, \mu_1)Z_f(\vec{a}, \vec{Y}, \mu_2)}{D(\vec{a}, \vec{Y})} = \frac{(\mu_1 - a)(\mu_2 - a)}{2iP}.$$

(4.11)

Level 2: The chain vector on the second level has the form

$$|2\rangle^{V\otimes\mathcal{H}} = |2\rangle^V + |1\rangle^V|1\rangle^{\mathcal{H}} + |2\rangle^{\mathcal{H}} = \left[\beta_2^V L_{-2} + \beta_{1,1}^V L_{-1}^2 + \beta_2^{\mathcal{H}} a_{-2} + \beta_{1,1}^{\mathcal{H}} a_{-1}^2 + \beta_1^V \beta_1^{\mathcal{H}} L_{-1} a_{-1}\right]|P\rangle,$$

(4.12)

where the explicit formulas for the coefficients on the second level:

$$\beta_2^V = -\frac{(\Delta - \Delta^2 - \Delta_1 - 2\Delta\Delta_1 + 3\Delta_1^2 - \Delta_2 - 2\Delta\Delta_2 - 6\Delta_1\Delta_2 + 3\Delta_2^2)}{(c - 10\Delta + 2c\Delta + 16\Delta^2)},$$
$$\beta_{1,1}^V = \frac{\Delta + 2\Delta_1 - \Delta_2}{6\Delta} + \frac{(c + 8\Delta)(\Delta - \Delta^2 - \Delta_1 - 2\Delta\Delta_1 + 3\Delta_1^2 - \Delta_2 - 2\Delta\Delta_2 - 6\Delta_1\Delta_2 + 3\Delta_2^2)}{12\Delta(c - 10\Delta + 2c\Delta + 16\Delta^2)},$$
$$\beta_2^{\mathcal{H}} = i\lambda_1,$$
$$\beta_{1,1}^{\mathcal{H}} = -2\lambda_1^2.$$

(4.13)

$N = 2$ basis vectors are

$$|\{2\}, \varnothing\rangle = \hat{J}_{\{2\}}(a_k - b_k)\hat{J}_\varnothing(a_k + b_k)|P\rangle = [(a_{-2} - b_{-2}) + (a_{-1} - b_{-1})^2]|P\rangle,$$
$$|\{1,1\}, \varnothing\rangle = \hat{J}_{\{1,1\}}(a_k - b_k)\hat{J}_\varnothing(a_k + b_k)|P\rangle = [(a_{-2} - b_{-2}) - (a_{-1} - b_{-1})^2]|P\rangle,$$
$$|\varnothing\{2\}\rangle = \hat{J}_\varnothing(a_k - b_k)\hat{J}_{\{2\}}(a_k + b_k)|P\rangle = [(a_{-2} + b_{-2}) + (a_{-1} + b_{-1})^2]|P\rangle,$$
$$|\varnothing\{1,1\}\rangle = \hat{J}_\varnothing(a_k - b_k)\hat{J}_{\{1,1\}}(a_k + b_k)|P\rangle = [(a_{-2} + b_{-2}) - (a_{-1} + b_{-1})^2]|P\rangle,$$
$$|\{1\}, \{1\}\rangle = \hat{J}_{\{1\}}(a_k - b_k)\hat{J}_{\{1\}}(a_k + b_k)|P\rangle = (a_{-1}^2 - b_{-1}^2)|P\rangle.$$

(4.14)

Now the scalar products on the level two can be easily calculated

$$\langle\{2\},\varnothing|2\rangle^{\mathsf{V}\otimes\mathcal{H}} = -\frac{(P-\lambda_1-\lambda_2)(i+P-\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(i+P-\lambda_1+\lambda_2)}{2P(i+2P)},$$

$$\langle\{1,1\},\varnothing|2\rangle^{\mathsf{V}\otimes\mathcal{H}} = -\frac{((P-\lambda_1-\lambda_2)(-i+P-\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(-i+P-\lambda_1+\lambda_2)}{2P(-i+2P)},$$

$$\langle\varnothing,\{2\}|2\rangle^{\mathsf{V}\otimes\mathcal{H}} = -\frac{(P+\lambda_1-\lambda_2)(-i+P+\lambda_1-\lambda_2)(P+\lambda_1+\lambda_2)(-i+P+\lambda_1+\lambda_2)}{2P(-i+2P)},$$

$$\langle\varnothing,\{1,1\}|2\rangle^{\mathsf{V}\otimes\mathcal{H}} = \frac{(P+\lambda_1-\lambda_2)(i+P+\lambda_1-\lambda_2)(P+\lambda_1+\lambda_2)(i+P+\lambda_1+\lambda_2)}{2P(i+2P)},$$

$$\langle\{1\},\{1\}|2\rangle^{\mathsf{V}\otimes\mathcal{H}} = \frac{(P-\lambda_1-\lambda_2)(P+\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(P+\lambda_1+\lambda_2)}{1+4P^2}.$$

$$(4.15)$$

Again we can check that this coincides with r.h.s. of (4.5), as it follows from the explicit expressions of $Z_f(a,Y,\mu_1)$, $Z_f(a,Y,\mu_2)$ and $D(a)$ for this case given in Appendix A. The details of the similar computations for the third level are collected in Appendix C. All these checks confirm our conjecture about the decomposition of the chain vectors (4.3) with the Necrasov's coefficients (4.5) in the orthogonal basis (4.2).

5   Conclusions

The vectors of the orthogonal basis are expected [14, 15] to be eigenstates of some commutative subalgebra of the universal enveloping algebra of $\mathcal{A} = Vir \otimes \mathcal{H}$. (The elements of this subalgera are called the Integrals of motion.) First few Integral of motion have been found in [15] explicitly. It is not difficult to check that in the case $c = 1$ (or $Q = 0$) each of them can be transformed to the sum of two terms. The first one depends only on $a_k + b_k$ and the second one depends on $a_k - b_k$. These terms are nothing but the Integrals of motion of the two Calogero models expressed in terms of Heisenberg generators. As it is known, the common eigenstates of the Integrals of motion of the Calogero model are the Jack polynomials. Therefore this fact is an additional confirmation of our conjecture about the factorization of the vectors $|Y_1, Y_2\rangle$ in the case $c = 1$ into the product of two Jack polynomials. In general case, i.e. if the central charge c is not equal 1, the explicit form of the vectors $|Y_1, Y_2\rangle$ is still an open problem.

Acknowledgments

Appendix A. AGT relations and Nekrasov partition functions

Let $Y = (\lambda_1 \geq \lambda_2 \geq \dots)$ be a Young tableau, where λ_i is the height of i-the column and is equal zero if i is larger than the width of the tableau. Let $Y^T = (\lambda_1' \geq \lambda_2' \geq \cdots)$ be its transposed. For a box s at the coordinate (i, j) a function $\phi(a, s)$ and its arm-length $A_Y(s)$ and leg-length $L_Y(s)$ (see figure 1) with respect to the tableau Y are defined as

$$\phi(a, s) = a + b(i - 1) + b^{-1}(j - 1), \tag{A.1}$$

$$A_Y(s) = \lambda_i - j, \qquad L_Y(s) = \lambda_j' - i. \tag{A.2}$$

If the box s is outside the tableau, $A_Y(s)$ and $L_Y(s)$ are negative.

In the example on the Figure 1:

$$\begin{aligned} A_Y(s) &= \text{number of unfilled circles}, \\ L_Y(s) &= \text{number of filled circles}. \end{aligned} \tag{A.3}$$
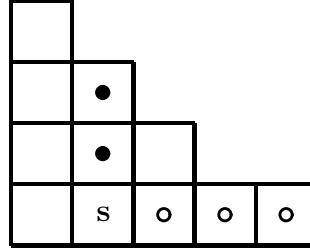

Figure 1: Young diagram, $A_Y(s)$ and $L_Y(s)$.

For two Young tableaux $\vec{Y} = (Y_1, Y_2)$ and the vector $\vec{a} = (a_1, a_2)$ the function E is defined as

$$E\big(a_i - a_j, Y_i, Y_j \big| s\big) = (a_i - a_j) - b A_{Y_j}(s) + b^{-1}(L_{Y_i}(s) + 1) \tag{A.4}$$

The explicit form of the functions $Z_{\mathsf{f}}(\vec{a}, \vec{Y}, \mu)$, $Z_{\mathsf{af}}(\vec{a}, \vec{Y})$ and $Z_{\mathsf{vec}}(\vec{a}, \vec{Y}, \mu)$ are

$$Z_{\mathsf{f}}(\vec{a}, \vec{Y}, \mu) = \prod_{i=1}^{2} \prod_{s \in Y_i} (\phi(a_i, s) - \mu + Q), \tag{A.5}$$

$$Z_{\mathsf{af}}(\vec{a}, \vec{Y}, \mu) = \prod_{i=1}^{2} \prod_{s \in Y_i} (\phi(a_i, s) + \mu) \tag{A.6}$$

and

$$Z_{\mathsf{vec}}(\vec{a}, \vec{Y}) = D(\vec{a}, \vec{Y}) \bar{D}(\vec{a}, \vec{Y}), \tag{A.7}$$

where the auxiliary functions

$$\begin{aligned} D(\vec{a}, \vec{Y}) &= \prod_{i,j=1}^{2} \prod_{s \in Y_i} E\big(a_i - a_j, Y_i, Y_j \big| s\big), \\ \bar{D}(\vec{a}, \vec{Y}) &= \prod_{i,j=1}^{2} \prod_{s \in Y_i} (Q - E\big(a_i - a_j, Y_i, Y_j \big| s\big)). \end{aligned} \tag{A.8}$$

Appendix B. Jack polynomials

In this appendix we briefly review the main properties of Jack symmetric functions. For more details see for example [17].

A partition of some natural n is a nonincreasing finite sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ such that the module $|\lambda| = \sum_i \lambda_i = n$. Another possible representation of the partition $\lambda = (1^{m_1} 2^{m_2} \dots)$, where m_k is a number of $\lambda_i = k$. Number of nonzero entries in λ is called length of λ and denoted by $l(\lambda)$. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ corresponds to a Young diagram such that the number of boxes in the ith column is λ_i. For two partitions λ and μ, if $|\lambda| = |\mu|$ and

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \qquad \text{for all } i. \tag{B.1}$$

one says that $\lambda \geq \mu$. This defines a dominance order on the set of partitions. Let λ be a partition with $l(\lambda) \leq N$. The monomial symmetric polynomial m_λ is defined as follows

$$m_\lambda(x_1, \dots, x_N) = \sum_{\sigma(\lambda_i)} x_1^{\lambda_{\sigma(1)}} \cdots x_N^{\lambda_{\sigma(N)}}, \tag{B.2}$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ runs over all distinct permutations of $(\lambda_1, \lambda_2, \dots, \lambda_N)$. The monomial symmetric functions form a basis in the space of symmetric functions.

The nth power sums are related to the monomial polynomials $m_{(n)}$ as

$$p_n = \sum x_i^n = m_{(n)}. \tag{B.3}$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, one can define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$. Then $\{p_\lambda\}$ also form a basis in the space of symmetric functions. The inner product in $\{p_\lambda\}$ can be defined as

$$\langle p_\lambda, p_\mu \rangle = \alpha^{l(\lambda)} \prod k^{m_k} m_k! \delta_{\lambda\mu}, \tag{B.4}$$

where $\lambda = (1^{m_1} 2^{m_2} \cdots)$ and α is some positive real number.

For each partition λ, there is a unique symmetric polynomial P_λ^α satisfying

$$P_\lambda^\alpha = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu}^{(\alpha)} m_\mu,$$
$$\langle P_\lambda^\alpha, P_\mu^\alpha \rangle = 0 \qquad \text{if} \qquad \lambda \neq \mu, \tag{B.5}$$

where $u_{\lambda\mu}^{(\alpha)}$ are some appropriate coefficients. The polynomials P_λ^α are called the Jack symmetric polynomials. The uniqueness follows from the explicit construction of $\{P_\lambda^\alpha\}$ by means of orthogonalization procedure. The nontrivial point here is that the summation runs over μ which is smaller than λ with respect to the dominance order.

In a different (so called "integral") normalization the Jack polynomials are denoted as $J_\lambda^{(\alpha)}$:

$$J_\lambda^{(\alpha)} = c_\lambda(\alpha) P_\lambda^\alpha, \qquad c_\lambda(\alpha) = \prod_{s \in Y} (\alpha A_Y(s) + L_Y(s) + 1), \tag{B.6}$$

where "$s \in Y$" means that s is a box of the Young tableau Y corresponding to λ and for "arm" and "leg" definitions see Appendix A.

We give the explicit form of Jack symmetric functions up to degree 3:

$$
\begin{aligned}
J_{\{1\}}^{(\alpha)} &= p_1, \\
J_{\{2\}}^{(\alpha)} &= \alpha p_2 + p_1^2, \\
J_{\{1,1\}}^{(\alpha)} &= -p_2 + p_1^2, \\
J_{\{3\}}^{(\alpha)} &= 2\alpha^2 p_3 + 3\alpha p_2 p_1 + p_1^3, \\
J_{\{1,2\}}^{(\alpha)} &= -\alpha p_3 + (\alpha - 1)p_2 p_1 + p_1^3, \\
J_{\{1,1,1\}}^{(\alpha)} &= 2p_3 - 3p_2 p_1 + p_1^3.
\end{aligned}
\tag{B.7}
$$

Appendix C. Level 3 computations

On the third level the mixed chain vector is

$$
|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = |3\rangle^{\mathsf{V}} + |2\rangle^{\mathsf{V}}|1\rangle^{\mathcal{H}} + |1\rangle^{\mathsf{V}}|2\rangle^{\mathcal{H}} + |3\rangle^{\mathcal{H}} =
$$

$$
\Bigg[\beta_3^{\mathsf{V}} L_{-3} + \beta_{1,2}^{\mathsf{V}} L_{-1} L_{-2} + \beta_{1,1,1}^{\mathsf{V}} L_{-1}^3 + + \beta_3^{\mathcal{H}} a_{-3} + \beta_{1,2}^{\mathcal{H}} a_{-1} a_{-2} + \beta_{1,1,1}^{\mathcal{H}} a_{-1}^3 +
$$

$$
+ \beta_2^{\mathsf{V}} \beta_1^{\mathcal{H}} L_{-2} a_{-1} + \beta_1^{\mathsf{V}} \beta_2^{\mathcal{H}} L_{-1} a_{-2} \Bigg] |P\rangle.
\tag{C.1}
$$

The coefficients of the Virasoro chain vector of the 3d level:

$$
\beta_3^{\mathsf{V}} = \frac{\Delta\Delta_1 - \Delta_1^2 - \Delta\Delta_1^2 + \Delta_1^3 - \Delta\Delta_2 - 3\Delta_1^2\Delta_2 + \Delta_2^2 + \Delta\Delta_2^2 + 3\Delta_1\Delta_2^2 - \Delta_2^3}{2\Delta(2 + c - 7\Delta + c\Delta + 3\Delta^2)},
$$

$$
\beta_{1,2}^{\mathsf{V}} = \frac{12\Delta^2 - 12\Delta^3 - 12\Delta\Delta_1 - 36\Delta^2\Delta_1 + 48\Delta\Delta_1^2 - 12\Delta\Delta_2 - 12\Delta^2\Delta_2 - 72\Delta\Delta_1\Delta_2 + 24\Delta\Delta_2^2}{-24c\Delta + 240\Delta^2 - 48c\Delta^2 - 384\Delta^3}
$$

$$
+ \frac{(-24c\Delta + 168\Delta^2 - 72c\Delta^2 - 216\Delta^3)}{2\Delta(2 + c - 7\Delta + c\Delta + 3\Delta^2)} \times
$$

$$
\times \frac{(-\Delta\Delta_1 + \Delta_1^2 + \Delta\Delta_1^2 - \Delta_1^3 + \Delta\Delta_2 + 3\Delta_1^2\Delta_2 - \Delta_2^2 - \Delta\Delta_2^2 - 3\Delta_1\Delta_2^2 + \Delta_2^3)}{(-24c\Delta + 240\Delta^2 - 48c\Delta^2 - 384\Delta^3)},
$$

$$
\beta_{1,1,1}^{\mathsf{V}} = \frac{\Delta + 3\Delta_1 - \Delta_2}{24\Delta} + \frac{c + 8\Delta}{12\Delta} \times
$$

$$
\times \Bigg[\frac{-12\Delta^2 + 12\Delta^3 + 12\Delta\Delta_1 + 36\Delta^2\Delta_1 - 48\Delta\Delta_1^2 + 12\Delta\Delta_2 + 12\Delta^2\Delta_2 + 72\Delta\Delta_1\Delta_2 - 24\Delta\Delta_2^2}{-24c\Delta + 240\Delta^2 - 48c\Delta^2 - 384\Delta^3} -
$$

$$
- \frac{(-24c\Delta + 168\Delta^2 - 72c\Delta^2 - 216\Delta^3)}{2\Delta(2 + c - 7\Delta + c\Delta + 3\Delta^2)} \times
$$

$$
\times \frac{(-\Delta\Delta_1 + \Delta_1^2 + \Delta\Delta_1^2 - \Delta_1^3 + \Delta\Delta_2 + 3\Delta_1^2\Delta_2 - \Delta_2^2 - \Delta\Delta_2^2 - 3\Delta_1\Delta_2^2 + \Delta_2^3)}{(-24c\Delta + 240\Delta^2 - 48c\Delta^2 - 384\Delta^3)} \Bigg] +
$$

$$
+ \frac{(c + 3\Delta)(-\Delta\Delta_1 + \Delta_1^2 + \Delta\Delta_1^2 - \Delta_1^3 + \Delta\Delta_2 + 3\Delta_1^2\Delta_2 - \Delta_2^2 - \Delta\Delta_2^2 - 3\Delta_1\Delta_2^2 + \Delta_2^3)}{24\Delta^2(2 + c - 7\Delta + c\Delta + 3\Delta^2)}.
\tag{C.2}
$$

The coefficients of the Heisenberg chain vector of the 3d level:

$$\beta_3^{\mathcal{H}} = \frac{2i\lambda_1}{3},$$
$$\beta_{1,2}^{\mathcal{H}} = -2i\lambda_1^2,$$
$$\beta_{1,2}^{\mathcal{H}} = -\frac{4i\lambda_1^3}{3}.$$

(C.3)

On the other side there exist 10 vectors $|Y_1, Y_2\rangle$ of the orthogonal basis on the third level:

$$|\{3\}, \varnothing\rangle = \hat{J}_{\{3\}}(a_k - b_k)\hat{J}_{\varnothing}(a_k + b_k)|P\rangle = [2(a_3 - b_3) + 3(a_2 - b_2)(a_1 - b_1) + (a_1 - b_1)^3]|P\rangle,$$
$$|\{1,2\}, \varnothing\rangle = \hat{J}_{\{1,2\}}(a_k - b_k)J_{\varnothing}(a_k + b_k)|P\rangle = [-(a_3 - b_3) + (a_1 - b_1)^3]|P\rangle,$$
$$|\{1,1,1\}, \varnothing\rangle = \hat{J}_{\{1,1,1\}}(a_k - b_k)J_{\varnothing}(a_k + b_k)|P\rangle = [2(a_3 - b_3) - 3(a_2 - b_2)(a_1 - b_1) + (a_1 - b_1)^3]|P\rangle,$$
$$|\varnothing, \{3\}\rangle = \hat{J}_{\varnothing}(a_k - b_k)\hat{J}_{\{3\}}(a_k + b_k)|P\rangle = [2(a_3 + b_3) + 3(a_2 + b_2)(a_1 + b_1) + (a_1 + b_1)^3]|P\rangle,$$
$$|\varnothing, \{1,2\}\rangle = \hat{J}_{\varnothing}(a_k - b_k)\hat{J}_{\{1,2\}}(a_k + b_k)|P\rangle = [-(a_3 + b_3) + (a_1 + b_1)^3]|P\rangle,$$
$$|\varnothing, \{1,1,1\}\rangle = \hat{J}_{\varnothing}(a_k - b_k)\hat{J}_{\{1,1,1\}}(a_k + b_k)|P\rangle = [2(a_3 + b_3) - 3(a_2 + b_2)(a_1 + b_1) + (a_1 + b_1)^3]|P\rangle,$$
$$|\{2\}, \{1\}\rangle = \hat{J}_{\{2\}}(a_k - b_k)\hat{J}_{\{1\}}(a_k + b_k)|P\rangle = [(a_2 - b_2) + (a_1 - b_1)^2](a_1 + b_1)|P\rangle,$$
$$|\{1,1\}, \{1\}\rangle = \hat{J}_{\{1,1\}}(a_k - b_k)\hat{J}_{\{1\}}(a_k + b_k)|P\rangle = [-(a_2 - b_2) + (a_1 - b_1)^2](a_1 + b_1)|P\rangle,$$
$$|\{1\}, \{2\}\rangle = \hat{J}_{\{1\}}(a_k - b_k)\hat{J}_{\{2\}}(a_k + b_k)|P\rangle = (a_1 - b_1)[(a_2 + b_2) + (a_1 + b_1)^2]|P\rangle,$$
$$|\{1\}, \{1,1\}\rangle = \hat{J}_{\{1\}}(a_k - b_k)\hat{J}_{\{1,1\}}(a_k + b_k)|P\rangle = (a_1 - b_1)[-(a_2 + b_2) + (a_1 + b_1)^2]|P\rangle.$$

(C.4)

With the matrix of scalar products $M = \langle a_{k_1} \cdots b_{m_1} \cdots | a_{-l_1} \cdots L_{-n_1} \cdots \rangle$ for $\sum k_i + \sum m_i = \sum l_i + \sum n_i = 3$

$$M = \begin{pmatrix}
3iP & 6iP & 6iP & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 - 2P^2 & -6P^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3iP}{2} & -6iP^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & iP & iP & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -P^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & iP & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{iP}{2}
\end{pmatrix},$$

(C.5)

one can find for the scalar products of the vectors $|Y_1, Y_2\rangle$ with the chain vector $|3\rangle^{\mathsf{V}\otimes\mathcal{H}}$ the following answers:

$$\langle\{3\},\varnothing|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (4iP(-i+P)(-i+2P))^{-1}(P+\lambda_1-\lambda_2)(-i+P+\lambda_1-\lambda_2)$$
$$\times\,(-2i+P+\lambda_1-\lambda_2)(P+\lambda_1+\lambda_2)(-i+P+\lambda_1+\lambda_2)(-2i+P+\lambda_1+\lambda_2),$$

$$\langle\{1,2\},\varnothing|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (2iP(1+4P^2))^{-1}(P+\lambda_1-\lambda_2)(P+\lambda_1+\lambda_2)$$
$$\times\,(1+P^2+2P\lambda_1+\lambda_1^2-2P\lambda_2-2\lambda_1\lambda_2+\lambda_2^2)$$
$$\times\,(1+P^2+2P\lambda_1+\lambda_1^2+2P\lambda_2+2\lambda_1\lambda_2+\lambda_2^2),$$

$$\langle\{1,1,1\},\varnothing|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (4(-1-iP)P(-i+2P))^{-1}(P-\lambda_1-\lambda_2)(-i+P-\lambda_1-\lambda_2)$$
$$\times\,(-2i+P-\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)\,(-i+P-\lambda_1+\lambda_2)(-2i+P-\lambda_1+\lambda_2),$$

$$\langle\varnothing,\{3\}|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (-4iP(i+P)(i+2P))^{-1}(P-\lambda_1-\lambda_2)(i+P-\lambda_1-\lambda_2)$$
$$\times\,(2i+P-\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(i+P-\lambda_1+\lambda_2)(2i+P-\lambda_1+\lambda_2),$$

$$\langle\varnothing,\{1,2\}|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (-2iP(1+4P^2))^{-1}(P-\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)$$
$$\times\,(1+P^2-2P\lambda_1+\lambda_1^2+2P\lambda_2-2\lambda_1\lambda_2+\lambda_2^2)$$
$$\times\,(1+P^2-2P\lambda_1+\lambda_1^2-2P\lambda_2+2\lambda_1\lambda_2+\lambda_2^2),$$

$$\langle\varnothing,\{1,1,1\}|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (4(-1-iP)P(-i+2P))^{-1}(P-\lambda_1-\lambda_2)(-i+P-\lambda_1-\lambda_2)$$
$$\times\,(-2i+P-\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(-i+P-\lambda_1+\lambda_2)(-2i+P-\lambda_1+\lambda_2),$$

$$\langle\{2\},\{1\}|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (4(-1-iP)P(i+2P))^{-1}(P-\lambda_1-\lambda_2)(P+\lambda_1-\lambda_2)$$
$$\times\,(-i+P+\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(P+\lambda_1+\lambda_2)(-i+P+\lambda_1+\lambda_2),$$

$$\langle\{1,1\},\{1\}|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (4(-1-iP)P(i+2P))^{-1}(P-\lambda_1-\lambda_2)(P+\lambda_1-\lambda_2)$$
$$\times\,(i+P+\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(P+\lambda_1+\lambda_2)(i+P+\lambda_1+\lambda_2),$$

$$\langle\{1\},\{2\}|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (4(-1+iP)P(-i+2P))^{-1}(P-\lambda_1-\lambda_2)(i+P-\lambda_1-\lambda_2)$$
$$\times\,(P+\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(i+P-\lambda_1+\lambda_2)(P+\lambda_1+\lambda_2),$$

$$\langle\{1\},\{1,1\}|3\rangle^{\mathsf{V}\otimes\mathcal{H}} = (4(-1+2iP)P(-i+P))^{-1}(P-\lambda_1-\lambda_2)(-i+P-\lambda_1-\lambda_2)$$
$$\times\,(P+\lambda_1-\lambda_2)(P-\lambda_1+\lambda_2)(-i+P-\lambda_1+\lambda_2)(P+\lambda_1+\lambda_2).$$

These expressions coincide with the Nekrasov's coefficients (4.5).

References

[1] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, Lett. Math. Phys. **91** (2010) 167–197, [arXiv:0906.3219]

[2] N. A. Nekrasov, *Seiberg-Witten Prepotential From Instanton Counting*, Adv. Theor. Math. Phys. **7** (2004) 831–864, [hep-th/0206161];
R. Flume and R. Poghossian, *An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, Int. J. Mod. Phys. **A18** (2003) 2541, [hep-th/0208176]

[3] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B241** (1984) 333–380.

[4] Al.Zamolodchikov, *Conformal Symmetry In Two-Dimensions: An Explicit Recurrence Formula For The Conformal Partial Wave Amplitude*, Commun.Math.Phys., 96 (1984) 419;
Conformal symmetry in two-dimensional space: Recursion representation of conformal block, Theor. Math. Phys. 73 (1987) 1088

[5] R. Poghossian, *Recursion relations in CFT and $\mathcal{N} = 2$ SYM theory*, JHEP **0912**, 038 (2009), arXiv:0909.3412[hep-th]

[6] L. Hadasz, Z. Jaskolski and P. Suchanek, *Recursive representation of the torus 1-point conformal block*, arXiv:0911.2353[hep-th];
Proving the AGT relation for $N_f = 0, 1, 2$ antifundamentals, arXiv:1004.1841[hep-th]

[7] V. Fateev and A. Litvinov, *On AGT conjecture*, JHEP **1002**, 014 (2010), arXiv:0912.0504[hep-th]

[8] V. S. Dotsenko and V. A. Fateev, *Conformal algebra and multipoint correlation functions in 2d statistical models*, Nucl. Phys. **B240** (1984) 312; *Four point correlation functions and the operator algebra in the two-dimensional conformal invariant theories with the central charge $c < 1$*, Nucl. Phys. **B251** (1985) 691

[9] R. Dijkgraaf and C. Vafa, *Toda theories,Matrix Models,Topological Strings and $N = 2$ Gauge Systems*, arXiv:0909.2453[hep-th]

[10] T. Eguchi and K. Maruyoshi, *Penner Type Matrix Model and Seiberg-Witten Theory*, JHEP **1002**, 022 (2010), arXiv:0911.4797[hep-th];
Seiberg-Witten theory, matrix model and AGT relation, arXiv:1006.0828[hep-th]

[11] H. Itoyama and T. Oota, *Method of Generating q-Expansion Coefficients for Conformal Block and $\mathcal{N} = 2$ Nekrasov Function by β-Deformed Matrix Model*, arXiv:1003.2929[hep-th]

[12] A. Mironov, A. Morozov, Sh. Shakirov, *Towards a proof of AGT conjecture by methods of matrix models*, arXiv:1011.5629[hep-th] *A direct proof of AGT conjecture at $\beta = 1$*, arXiv:1012.3137[hep-th]

[13] L. F. Alday and Y. Tachikawa, *Affine $SL(2)$ conformal blocks from 4d gauge theories*, Lett.Math.Phys. **94** (2010) 87–114, arXiv:1005.4469[hep-th]

[14] B. L.Feigin, *The talks in Moscow Independent University and Landau Institute*, unpublished, (2010)

[15] V. Alba, V. Fateev, A. Litvinov and G. Tarnopolsky, *On combinatorial expansion of the conformal blocks arising from AGT conjecture*, arXiv:1012.1312[hep-th]

[16] E. Carlsson and A. Okounkov, *EXTS and vertex operators*, arXiv:0801.2565[math]

[17] R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. 77 (1989) 76–115.

[18] R. Sakamoto, J. Shiraishi, D. Arnaudon, L. Frappat, and E. Ragoucy, *Correspondence between conformal field theory and Calogero-Sutherland model*, Nucl. Phys. **B704**, (2005) 490–509, [`hep-th/0407267`]

[19] D. Gaiotto, $\mathcal{N} = 2$ *dualities*, arXiv:0904.2715[hep-th]

[20] B. L. Feigin and D. Fuks, *Verma modules over Virasoro algebra*. Moscow preprint, (1983).

[21] A. B.Zamolodchikov and Al. Zamolodchikov, *Structure Constants and Conformal Bootstrap in Liouville Field Theory*, Nucl. Phys. **B477**, (1996) 577-605, arXiv:hep-th/95063136